Q1a

1a

Note: For this question ensure you are working in degrees.
A wave tank is used to simulate the sea at high tide.
At a certain point along the tank the height of water is measured relative to the calm water level which has a height of 0 cm.
The height of water in the tank is modelled by the function

$$h(t) = 12\cos(20t)^{\circ} \qquad t \geq 0$$

where h cm is the height of water and t seconds is the time after the peak of the first wave passes the measuring point.

(a) Sketch a graph of h against t for $0 \leq t \leq 54$.

[3]

(b) (i) What is the maximum height the water reaches according to the model?
(ii) How frequent are the waves generated by the tank?
(iii) How often is the water at its calm level?
(iv) When will the peak of the 12th wave pass the measuring point?

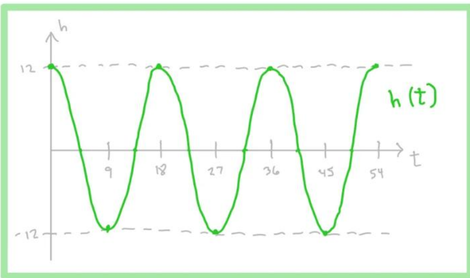
[4]

(c) Comment on the suitability of using this model to simulate actual sea waves.

[1]

a) $0 \leq t \leq 54 \Rightarrow 0 \leq 20t \leq 1080$

$\frac{1080}{360} = 3 \Rightarrow 3$ complete cosine cycles



Q1b

1b

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[3]

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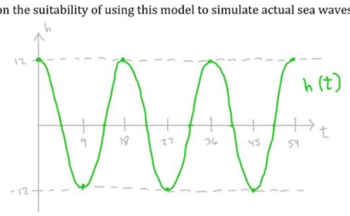
[4]

(c) Comment on the suitability of using this model to simulate actual sea waves.



[1]

b) (i) $\boxed{12 \text{ cm}}$

(ii) $\boxed{\text{One wave every 18 seconds}}$

(iii) $\boxed{\text{Every 9 seconds, starting at } t = 4.5}$

(iv) $11 \times 18 = 198 \Rightarrow \boxed{t = 198 \text{ seconds}}$

Be careful here! The 1st wave is at $t=0$.

Q1c

1c

Note: For this question ensure you are working in degrees.
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where h cm is the height of water and t seconds is the time after the peak of the first wave passes the measuring point.

(a) Sketch a graph of h against t for $0 \leq t \leq 54$.

[3]

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(iii) How often is the water at its calm level?
(iv) When will the peak of the 12th wave pass the measuring point?

[4]

(c) Comment on the suitability of using this model to simulate actual sea waves.

[1]

c) Real waves aren't all exactly the same (their heights are different, for example). The model's waves are all exactly the same, so the model is not entirely realistic.

Q2a

2a

The function

$$t(T) = 25\ln\left(\frac{98.6 - R}{T - R}\right) \qquad R < T \leq 98.6$$

is used as a model to estimate the time, t hours, since the death of a human body.
T °F is the temperature of the body at a given time.
R °F is the ambient (surrounding) temperature, assumed to be constant.

(a) (i) If a body records a temperature of 81 °F at an ambient temperature of 70 °F, estimate how long ago the person died.

(ii) A suspected murder victim's body was discovered 15 hours after the victim died. Assuming the victim was found in a room of fixed temperature 70 °F, what temperature should the body have registered when it was discovered?

[5]

(b) (i) What does the value 98.6 in the model represent?
(ii) Describe a problem with using the model for a body temperature reading that is close to the ambient temperature.

[2]

a) (i) when $R = 70$ and $T = 81$,

$t = 25\ln\left(\frac{98.6-70}{81-70}\right) = 23.887786...$

$\boxed{t = 23.9 \text{ hours (to 1 d.p.)}}$

(ii) when $R = 70$ and $t = 15$,

$25\ln\left(\frac{98.6-70}{T-70}\right) = 15$

$\ln\left(\frac{28.6}{T-70}\right) = \frac{15}{25} = 0.6$

$\frac{28.6}{T-70} = e^{0.6}$ } take exponential of both sides

$28.6 = (T-70)e^{0.6}$

$T = \frac{28.6}{e^{0.6}} + 70 = 85.696012...$

$\boxed{T = 85.7°F \text{ (to 1 d.p.)}}$

Q2b

2b

The function

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[5]

(b) (i) What does the value 98.6 in the model represent?

(ii) Describe a problem with using the model for a body temperature reading that is close to the ambient temperature.

[2]

b) (i) When $T = 98.6$,

$t = 25\ln\left(\frac{98.6-R}{98.6-R}\right) = 25\ln(1) = 0$

98.6° F is the temperature at the time of death.

(ii) As T gets close to R, $\frac{98.6-R}{T-R}$ gets bigger and bigger, and so $t = 25\ln\left(\frac{98.6-R}{T-R}\right)$ gets bigger and bigger. In reality, we expect the temperature of the body to reach the ambient temperature in a finite amount of time.

Q3

3

A gardener is modelling the number of hours of daylight his allotment receives at different times of the year using the function

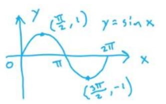
$$h(t) = 12 - a\sin\left(\frac{2\pi t}{365}\right) \qquad t \ge 0$$

where h is the number of hours of daylight on a given day, t is the time measured in whole days, and a is a positive constant.

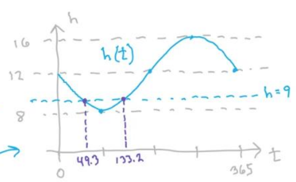
(i) Given that the maximum amount of daylight predicted by the model is 16 hours write down the value of a.

(ii) The gardener is also a keen golfer.
In order to have enough daylight to play golf after working in the garden there needs to be at least 9 hours daylight in the day.
On approximately how many days of the year can the gardener ***not*** play golf.

[5]



No need to sketch $h(t)$ to get the marks, but it can help with visualising the question



(i) h is at its maximum when $\sin\left(\frac{2\pi t}{365}\right) = -1$

Then $16 = 12 - a(-1) = 12 + a$ $\boxed{a = 4}$

(ii) $h = 9$ means

$9 = 12 - 4\sin\left(\frac{2\pi t}{365}\right)$

$4\sin\left(\frac{2\pi t}{365}\right) = 3 \implies \sin\left(\frac{2\pi t}{365}\right) = \frac{3}{4}$

$\frac{2\pi t}{365} = \sin^{-1}\left(\frac{3}{4}\right) = 0.848062\ldots$

or $\frac{2\pi t}{365} = \pi - \sin^{-1}\left(\frac{3}{4}\right) = 2.293530\ldots$ } by symmetry of sine function

So $t = 49.3$ or 133.2 (to 1 d.p.)

$t = 49 \implies$ long enough
$t = 50 \implies$ too short
$t = 133 \implies$ too short
$t = 134 \implies$ long enough

From $t = 50$ to $t = 133$ (inclusive) is 84 days

So the gardener cannot play golf on approximately 84 days.

Alternatively: $133.2 - 49.3 = 83.9$ days
$= 84$ (to nearest day)

Q4a

4a

The extremely rare Bouncing Unicorn has the ability to jump repeatedly with precision, such that both the height of the jump and the length of the jump remain constant.

The way in which Bouncing Unicorns jump can be modelled by the function

$$h(x) = |a \sin bx| \qquad x \geq 0$$

where x is the horizontal distance covered and h is the height, both measured in metres. a and b are positive constants.

(a) (i) Explain the meaning of the constant b in the context of the model.

(ii) A fully-grown Bouncing Unicorn jumps to a maximum height of 2.5 metres covering a horizontal distance of $\frac{2\pi}{3}$ metres in the process. Write down the values of a and b for a fully-grown Bouncing Unicorn.

[3]

(b) A fully-grown Bouncing Unicorn takes 3 seconds to complete one jump. Estimate the amount of time during a single jump that a Bouncing Unicorn spends 1.5 metres or more above the ground. State any assumptions you make for this question.

[5]

You don't need to sketch h(x) to get the marks, but it can be very helpful

a) (i) The unicorn completes a jump each time $h = 0$

$h = 0 \Rightarrow \sin bx = 0 \Rightarrow bx = 0, \pi, 2\pi$, etc.

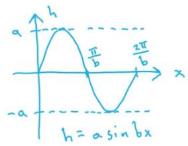
$bx = 0, \pi, 2\pi, \ldots \Rightarrow x = 0, \frac{\pi}{b}, \frac{2\pi}{b}, \ldots$

$\boxed{\frac{\pi}{b} \text{ metres is the length of a jump.}}$

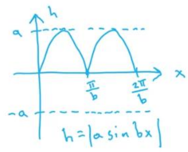
(ii) $\boxed{a = 2.5}$

$\frac{\pi}{b} = \frac{2\pi}{3} \Rightarrow \frac{1}{b} = \frac{2}{3}$

$\boxed{b = \frac{3}{2} = 1.5}$



$\Longrightarrow$



Q4b

4b

The extremely rare Bouncing Unicorn has the ability to jump repeatedly with precision, such that both the height of the jump and the length of the jump remain constant.

The way in which Bouncing Unicorns jump can be modelled by the function

$$h(x) = |a \sin bx| \qquad x \geq 0$$

where x is the horizontal distance covered and h is the height, both measured in metres. a and b are positive constants.

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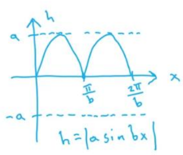

From part (a),
$a = 2.5$
$b = 1.5$

b) The x-coordinates of the first two points the unicorn is at 1.5 m height can be found by solving

$2.5 \sin(1.5x) = 1.5 \Rightarrow \sin(1.5x) = 0.6$

$1.5x = \sin^{-1}(0.6)$ or $\underbrace{\pi - \sin^{-1}(0.6)}_{\text{by symmetry of sine function}}$

$x_1 = \frac{\sin^{-1}(0.6)}{1.5} = 0.429000\ldots$

$x_2 = \frac{\pi - \sin^{-1}(0.6)}{1.5} = 1.665394\ldots$

If we assume the horizontal speed of the unicorn is constant, then the approximate time at or above 1.5 metres is

$\frac{x_2 - x_1}{2\pi/3} \times 3 = 1.771003\ldots$

$\boxed{1.77 \text{ seconds (3 s.f.)}}$

$\frac{x_2 - x_1}{2\pi/3}$ is the proportion of the horizontal distance of a single jump that is at or above a height of 1.5 metres

Q5a

5a

In a simple model of an investment account, the function

$$V(t) = I\left(1 + \frac{r}{100}\right)^t$$

is used where I is the initial amount invested, r % is the interest rate, and V is the value of the investment at the end of t years.

(a) (i) For an initial investment of £7500, find the interest rate if, after 5 years, the investment value is £8250.

(ii) Find the least number of years an initial amount of money would need to be invested at an interest rate of 5.6% in order for its value to triple.

[5]

(b) In 1998 the interest rate was 6.33%.
In 2019 the interest rate was 1.39%.
Assuming the interest rate remains constant from the initial time of investment, find roughly how many times greater the initial investment would have to be in 2019 compared to 1998 if the aim in both cases was to have an investment worth a million pounds 25 years later.

[3]

a) (i) when $V = 8250$, $I = 7500$, $t = 5$ then

$$7500\left(1 + \frac{r}{100}\right)^5 = 8250$$

$$\left(1 + \frac{r}{100}\right)^5 = \frac{8250}{7500}$$

$$1 + \frac{r}{100} = \left(\frac{8250}{7500}\right)^{1/5}$$

$$r = 100\left(\left(\frac{8250}{7500}\right)^{1/5} - 1\right) = 1.92448\ldots$$

$$\boxed{r = 1.92\,\% \text{ (3 s.f.)}}$$

(ii) when $V = 3I$ and $r = 5.6$, then

$$I(1.056)^t = 3I$$

$$(1.056)^t = 3 \qquad \ln(a^b) = b\ln a$$

$$\ln(1.056^t) = \ln 3$$

$$t\ln 1.056 = \ln 3$$

$$t = \frac{\ln 3}{\ln 1.056} = 20.162394\ldots$$

For the money to triple, the least number of whole years is 21.

Q5b

5b

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Assuming the interest rate remains constant from the initial time of investment, find roughly how many times greater the initial investment would have to be in 2019 compared to 1998 if the aim in both cases was to have an investment worth a million pounds 25 years later.

[3]

b) $1000000 = I_{1998}(1.0633)^{25}$

$$I_{1998} = \frac{1000000}{(1.0633)^{25}} = 215578.1065\ldots$$

$$1000000 = I_{2019}(1.0139)^{25}$$

$$I_{2019} = \frac{1000000}{(1.0139)^{25}} = 708144.5607\ldots$$

$$\frac{I_{2019}}{I_{1998}} = 3.284863\ldots$$

You would need an investment about 3.3 times greater in 2019.